ISSN: 2773-7055
Proceedings of 3rd International Research Conference
Institute of Technology, University of Moratuwa
December 18, 2024

A REVIEW ON FRACTIONAL CALCULUS AND ITS APPLICATIONS IN ENGINEERING

C. I. Perera*
Institute of Technology University of Moratuwa, Sri Lanka
pereracassian@gmail.com*

ABSTRACT: The main purpose of this research is to explore the evolution of fractional calculus from its inception to its diverse applications in science and engineering. The concept of calculus dates back to the work of Sir Isaac Newton, an English mathematician, and Gottfried Leibniz, a French mathematician who presented similar ideas using different notations. Initially, the order of differentiation was understood as an integer. However, as time progressed, mathematicians encountered a paradox: what if the order of a derivative were not an integer but a fraction? This question marked the beginning of fractional calculus.
Over the years, a lot of definitions were developed by renowned mathematicians such as Euler, Lagrange, and Laplace. In its early stages, fractional calculus was primarily theoretical with little application to practical problems. However, as the field matured, mathematicians began to apply this theory to practical situations. Today, fractional calculus has seen rapid development and application across various domains in science and engineering. Some applications are shortlisted as follows. In the field of computer and electrical engineering fractional calculus is used in noise filtering processes, particularly in echocardiographic imaging, to minimize noise interference. It also plays a crucial role in developing de-noising models in digital imaging. In mineral engineering, leaching column test is conducted by using new rate equation formed by fractional calculus. The Behaviour of Hydrological processes on Earth has been mathematically modelled using fractional calculus. In the field of quantum mechanics, the Schrödinger equation has been further improved using the fractional calculus theory in the formation of a new version of the fractional Schrödinger equation in the context of space-time.

Keywords: Fractional calculus, fractional derivative, fractional integral.

1. INTRODUCTION

The subject of calculus, one of the most important branches of mathematics, has its roots in problem-solving practices dating back to the Babylonian era. However, it was formally developed in the 17th century by two great mathematicians: Sir Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716). At that time, calculus was primarily divided into two main components: differentiation and integration, both of which were confined to integer orders. A significant question arose regarding the implications of a derivative order that was not an integer but a fraction. This inquiry was raised by the mathematician L'Hôpital, referencing the work of Leibniz (De Oliveira et al., 2014). Leibnitz's reply was that it may be a paradox. However, it would be very helpful in the future (Debnath, 2004). This was the starting point of Fractional Calculus. It means that 'fractional calculus 'was born to deal with the order of derivatives in non-integer.

From that time until 1900, there was slow progress in the development of theories on 'fractional calculus'. (De Oliveira et al., 2014). However, after Leibnitz, there were great mathematicians who applied worthy efforts to formulate theories on fractional calculus. For example, Euler in 1738, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Riemann in 1847, Green in 1859, Holmgren in 1865, Grunwald in 1867, Letnikov in 1868, Sonini in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, Weyl in 1919, Caputo in 1967, Khalil et.al in 2013 all contributed valuable efforts to the formulation of theories on fractional calculus (Lazarevic et al., 2014).

2. HISTORY OF FRACTIONAL CALCULUS

Let us examine the work done on fractional calculus by various mathematicians over the years.

2.1 From 1700 to 1750

2.1.1 Euler

In 1730, Euler derived a formula for $the\ n^{th}$ derivative of a power function as follows:

$$\frac{d^n x^m}{dx^n} = \text{m (m} - 1\text{) (m} - 2\text{)… (m – n +1)}x^{m-n}$$ ------- (1) (Dalir et al., 2010)

He later expressed it using Gamma functional notation:

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n} \text{------- (2)}$$

where Gamma function is defined as : $\Gamma(n) = \int_0^\infty e^{-t}\, t^{n-1} dt = (n-1)!$; $n \in R^+$

2.2 From 1801 to 1850

2.2.1 Lacroix

In 1819, Lacroix expressed the 'fractional derivative' of a power function assuming that it is similar to the n^{th} derivative of the power function y = x^m in 'classical calculus' where $\boldsymbol{m}$ is a positive integer, $\boldsymbol{m} \geq \boldsymbol{n}$ (Kimeu, 2009):

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n} \text{------ (3)}$$

2.2.2 Loiuville

In 1832, Loiuville formulated the **n**th derivative of an exponential function as follows:

$$D^n e^{ax} = a^n e^{ax} \text{ ------ (4)}$$

When a function $f(x)$ is given as a series of exponential terms, the fractional derivative of order ν of the function $f(x)$ was found by Loiuville as follows:

$$D^\nu f(x) = \sum_{n=0}^{\infty} c_n\, a_n^\nu e^{a_n x} \text{------- (5)} \quad \text{where } f(x) = \sum_{n=0}^{\infty} c_n\, e^{a_n x}$$

Debnath (2004) states this was his first definition on fractional derivative and his second definition was associated with Gamma functions.

$$D^\alpha x^{-\beta} = (-1)^\alpha \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} x^{-\alpha-\beta} \text{ ------ (6)}$$

(Debnath, 2004; Ross, 1977)

Where $\beta > 0$. If $\beta = 0$, the function become unity. Then, fractional derivative of unity (x^0) would not be equal to zero even though the derivative of unity in traditional calculus is zero (Lazarevic, 2014; Debnath, 2004). However, according to De Oliveira (2014), his next definition on fractional derivative of any function of order **α** in terms of integral is:

$$D^\alpha x^{-\beta} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} (x-\xi)^{-\alpha} f(\xi) d\xi \text{------- (7)}$$

Where $-\infty < x < \infty$.

Liouville was the first person to apply fractional derivative knowledge to solve differential equations in traditional calculus. (Ross, 1977)

2.2.3 Riemann

Riemann derived the following formula that shows how to find fractional integral of a function $f(x)$.

$$D_{c,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t) dt + \Psi(x) \text{------ (8)}$$

where $D_{c,x}^{-\alpha}$ denotes the fractional integral of order α while c, x are limits of integration (Lazarevic, 2014; Ross, 1977).

Here $\Psi(x)$ is referred to as the complementary function which was a term introduced by him. Later, a new version of this fractional integral was published omitting the complementary function (Ross, 1977) as mentioned below(Debnath, 2004; Kimeu, 2009).

$$D_{c,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t) dt \text{------ (9)}; \ \alpha > 0$$

This is known as the "Riemann – Loiuville definition of fractional integral" which is attributed solely to Riemann when c = 0, otherwise belongs to Loiuville if $\boldsymbol{c} = -\infty$ (Debnath, 2004).

2.2.4 Caputo

Caputo defined his own fractional derivative of a function as:

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}x}{(t-x)^{\alpha-n+1}} dx \text{ ------- } (10)$$

where $\alpha \epsilon$ [n-1, n), D_a^α is the fractional derivative operator, with **α** as the order and **a** as the lower limit of the integral included in the formula. (Khail et al., 2014)

2.3 The Time Period After 2000

2.3.1 Khalil et al.'s Definition for Fractional Derivative of Order α

Case 1: If 0 < α ≤1

Definition:

Given a function **f: [0, ∞)** → ℝ. Then the fractional derivative of ***f*** order $\boldsymbol{\alpha}$ is defined by

$$T\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \text{ ------ } (11)$$

This definition is true when **α** = 1. Because when $\boldsymbol{\alpha}$ = 1 it yields,

$$T\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon) - f(t)}{\varepsilon} \text{ ------- } (12)$$

This is the result of the first principle derivation of the first derivative of a function f in traditional calculus.

Case 2: If n < α ≤ n+1

Definition:

Let $\alpha \epsilon$ (n, n+1], and ***f*** be an ***n***-differentiable at ***t*** (>0). Then, the fractional derivative of order of $\boldsymbol{\alpha}$ is defined as:

$$T\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f^{(\lceil\alpha\rceil-1)}(t+\varepsilon t^{(\lceil\alpha-1\rceil)}) - f^{(\lceil\alpha\rceil-1)}(t)}{\varepsilon} \text{ ------- } (13)$$

2.3.2 Fractional Integral

Definition:

According to Khalil, et al. (2014), α- fractional integral of a function f (t) is as follows:

$$I_\alpha^a f(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx \text{ ------ } (14)$$

If the α- fractional derivative of the above integrand is determined, it could be proved that the final result would be ***f*** **(*t*)** confirming the accuracy of this method.

3. APPLICATIONS IN SCIENCE AND ENGINEERING

Some applications in science and engineering could be short listed as follows.

In the field of computer and electrical engineering, fractional calculus has been applied in noise filtering processes to reduce noise in echocardiographic imaging (Saadia et al., 2018). He et al. (2014) demonstrated how fractional-order differentiation is beneficial in developing de-noising models in digital imaging. In mineral engineering, leaching column tests are conducted using new rate equations formed by fractional calculus (Jaques et al., 2017). Research conducted by Zhang et al., (2017) shows the behavior of Hydrological processes on earth has been mathematically modelled using fractional calculus. Laskin (2017) illustrated the usefulness of fractional derivatives in formulating a new version of the fractional Schrödinger equation in the context of space-time within quantum mechanics. Recent applications of fractional calculus extend to dynamical systems including control theory, electrical circuits, viscoelasticity, electrochemistry, tracer in fluid flows, and model of neurons in biology.

4. CONCLUSION

This paper explores the definitions of fractional derivatives and fractional integrals of functions, highlighting the "Riemann–Liouville definition" and the "Caputo definition" as the most widely used. Initially, these concepts were not applied to practical situations. However, there has been rapid development in various scientific fields, including quantum mechanics, as well as in electrical, chemical, and civil engineering, through the application of fractional calculus. This approach addresses practical scenarios more effectively than classical calculus. Furthermore, applying fractional calculus to fractional partial differential equations (fPDEs) could significantly enhance research on the Earth's hydrosphere in the future.

5. REFERENCES

Dalir, M., Bashour, M. (2010). Applications of fractional calculus. *Applied Mathematics Sciences, 4*(21), 1021 – 1032.

Debnath, L. (2004) A brief historical introduction to fractional calculus. *International Journal of Mathematical Education in Science and Technology, 34*(4), 487-501, https://doi.org/10.1080 /0020739041000168657l.

De Oliveira1, E. C., Machado, J.A.T. (2014). A Review of Definitions for Fractional Derivatives and Integral. http://dx.doi.org/10.1155/2014/238459

He, N., Wang, J., Zhang, L., & Lu, K. (2014). An improved fractional-order differentiation model for image de-noising, Signal Processing, http://dx.doi.org/10.1016/j.sigpro.2014.08.025

Jaques, A., Da Silva, C., Duru, N., & Seal, T. (2017). Fractional calculus application for diffusion controlled leaching column testing. *Int. J. Miner. Process*, http://dx.doi.org/10.1016/ j.minpro.2017.09.007

Khalil, R., Al Horani, M., Yousuef, A., Sababheh, M., (2014). A new definition of fractional derivative.

Kimeu, Joseph M., (2009). Fractional Calculus: Definitions and Applications. *Masters Theses & Specialist Projects.*

Laskin, N., (2017). Time fractional quantum mechanics. http://dx.doi.org./10.1016/ j.chaos.2017.04.010

Lazarevic, M.P., Rapaic, M.R., Sekara, T.B. (2014) Introduction to fractional calculus with brief historical background. Retrieved from http://www.researchgate.net/publication/312137269

Ross, B., (1977). The development of fractional calculus 1695 – 1900. *Historia Mathematica*, *4*, 75 – 89.

Saadia, A., & Rashdib, A. (2018) Incorporating fractional calculus in echo-cardio graphic image de-noising. Retrieved from www.elsevier.com/locate/compeleceng.

Zhang, Y., Sun, H.G., Stowell, H. H., Zayernouri, M., & Hansen, S., E. (2017). A review of applications of fractional calculus in Earth system dynamics. *Chaos, Solitons and Fractals*.

http://dx.doi.org/10.1016/j.chaos.2017.03.051